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FINITE DEFLECTION AND SNAPPING OF THIN ELASTIC FLAT PANELS†

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The virtues of our predecessors are to be proud of.
Not to do so is shameful cowardice.
A. S. Pushkin

The problem of the non-linear behaviour and stability loss of bars and shells under finite deflection is discussed. The one-dimensional model, which plays a major part in the formulation of non-linear stability problems, is described in detail. In this connection, Ivan Grigor'yevich Bubnov's problem on the behaviour of a thin elastic cylindrical panel is analysed (the problem was solved by him in 1902 and was the first study on finite deflections of thin shells). The significance of this result in the theory of shells in its own right and from the perspective of the subsequent development of the theory is discussed. The importance of Bubnov's results in shell theory is pointed out. This also includes his solution of the non-linear behaviour of circular plates and plane elastic panels. This is why he can be regarded as a forerunner in formulating the equations of finite deflections of elastic thin-walled Foeppl–Kármán–Marguerre surfaces.

The development of the theory of elastic shells subject to finite deflection, especially in relation to the problem of stability, has been strongly influenced by results on the non-linear behaviour and snapping of flat plane bars. These results made it possible to establish a new mechanism of stability loss occurring for finite deflections because of a deflection jump, i.e. snapping, unlike stability loss for small deflections.

This process was first described by Timoshenko [1] (1925) in an article devoted to the loss of stability of a flat plane sinusoidal bimetal strip with clamped ends, heated uniformly over its thickness and length. It was established that because of the deflection and compression of the strip in the longitudinal direction it will snap towards the centre of curvature at temperature

$$T_- = (1+c)\left[\frac{3}{16}\frac{l^2}{Hh}(\alpha_2-\alpha_1)\right]^{-1} \tag{0.1}$$

where

$$c = \frac{2}{3\sqrt{3}}\frac{(1-m)^{3/2}}{m}, \quad m = \frac{h^2}{3H^2}$$

H is the elevation of the central axis of the strip, h and l are its thickness and length, and α_1 and α_2 are the coefficients of linear thermal expansion of the bimetal layers. If the temperature is subsequently reduced the strip will snap in the opposite direction at a temperature

$$T_+ = \frac{1-c}{1+c}T_- \tag{0.2}$$

In modern terminology, these can be called the upper and lower critical temperatures T_- and T_+. Timoshenko also obtained expressions for the deflections leading to snapping on the assumption that deflection varies along the axis as the sine function over a half-cycle. In this way a new mechanism of stability loss was described, which is well known from the practice of using bimetal elements in thermostats and plane-spherical perfume containers. In the paper under consideration attention was devoted only to bimetal elements in thermostats. An equivalent uniform bar was introduced in place of a bimetal one, and questions which would lead into a blind alley in the traditional approach were circumvented. The paper was noted by specialists working on bimetals, but completely ignored by those working on the theory of plates and shells.

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Ten years later in a paper entitled "Buckling of flat bars and slightly curved plates" [2]. Timoshenko considered a flat sinusoidal bar clamped at the ends subject to a uniform transverse pressure q parallel to the centre of curvature of the bar. He obtained the dependence of the deflection of the central axis of the bar on the transverse pressure and analysed the forms of equilibrium of the bar and their stability (the analysis below differs from Timoshenko's), determining the loads at which downwards snapping (q_-) upwards snapping (q_+) of the bar relative to the initial centre of curvature occurs. Since the deflection amplitude parameter was introduced in the approximate solution for a clamped bar subject to transverse load only, it became possible to extend this solution to other loadings by an appropriate choice of this parameter. For a bimetal bar the critical temperatures similar to (0.1) and (0.2) were given. Earlier solutions by Navier [3] (1926), Nadai [4] (1915), and Biezeno [5] (1929) for the case of a central point force acting on a flat bar clamped at the ends were mentioned. However, in these papers there is no description of the whole snapping process.

The mechanism of bar snapping established by Timoshenko and the critical loads q_- and q_+ found by him made it possible to look at the problem of the loss of stability in thin shells from the same point of view and to obtain the first results on the non-linear behaviour of a spherical shell subject to uniform radial pressure, and a circular cylindrical shell subject to uniform longitudinal compression. This was done by Kármán and Tsien in [6, 7] (1939–1941).

Timoshenko's results [2] were extended by Grigolyuk [8] (1951) to the case of the asymmetric snapping of a flat bar. A complete mechanical picture of the process was established. In [8] Bubnov's method of solving the finite deflection equation for a flat sinusoidal bar with clamped ends subject to a uniform pressure q and representing the deflection as a two-parameter function

$$w = w_0 \sin \pi\xi + w_1 \sin 2\pi\xi \quad (\xi = x / l)$$

was used to obtain the following relations

$$W_0^3 - 3W_0^2 + (4W_1^2 + 2 + m)W_0 - 4W_1^2 = \frac{4}{\pi^5} q^* m$$

$$W_1^3 + \left(\frac{1}{4} W_0^2 - \frac{1}{2} W_0 + m\right) W_1 = 0 \tag{0.3}$$

$$k^2 = -\frac{\pi^2}{m}\left(W_0^2 - 2W_0 + 4W_1\right)$$

$$W_0 = \frac{w_0}{H}, \ W_1 = \frac{w_1}{H}, \ q^* = \frac{ql^4}{EJH}, \ m = \frac{4J}{FH^2}, \ k^2 = -\frac{Nl^2}{EJ}$$

Here l is the bar length, x is the longitudinal coordinate, E is Young's modulus, J is the moment of inertia of the cross-section of the bar, H is the elevation of the bar, N is the longitudinal force, and F is the cross-section area of the bar.

This implies the following solution for the axially symmetric snapping of the bar with $W_1 = 0$

$$W_0^3 - 3W_0^2 + (2 + m)W_0 = \frac{4}{\pi^5} q^* m, \ k^2 = -\frac{\pi^2}{m}(W_0^2 - 2W_0)$$

$$q^{*2} - \frac{\pi^5}{2} q^* + \frac{\pi^4}{16}\left[mk^6 - (1 + 2m)\pi^2 k^4 + (2 + m)\pi^4 k^2\right] = 0$$

which yields the following expressions for the upper and lower critical loads q^*_- and q^*_+ obtained by Timoshenko [2]

$$q^*_{\mp} = \frac{\pi^4}{4}\left[1 \pm \frac{2\sqrt{3}}{9} \frac{(1 - m)^{3/2}}{m}\right]$$

and the corresponding deflections and longitudinal force parameters

$$W_{0_\mp} = 1 \mp \frac{\sqrt{3}}{3}\sqrt{1 - m}, \ W_{0_{1,2}} = 1 \pm 2\frac{\sqrt{3}}{3}\sqrt{1 - m}$$

$$k_\mp^2 = -\frac{\pi^2}{3}\left(1 + \frac{1}{m}\right), \ k_{1,2}^2 = -\frac{\pi^2}{3}\left(4 - \frac{1}{m}\right)$$

Snapping is possible when $0 < m \leqslant 1$ with $q^*_\mp \to \infty$ if $m = 0$. Graphs of $q^* = q^*(W_0)$ and $k^2 = k^2(q^*)$ are shown

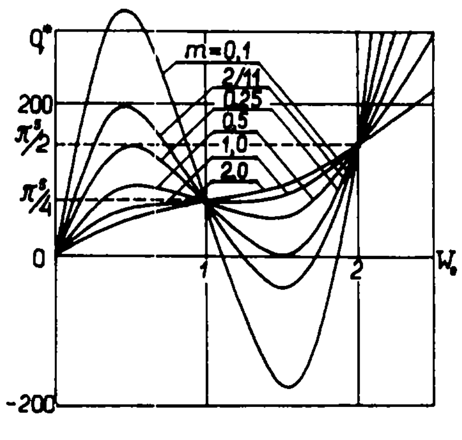

Fig. 1.

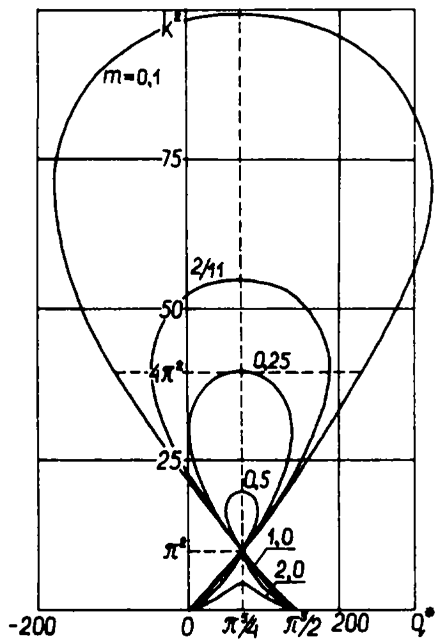

Fig. 2.

in Figs 1 and 2 for various values of the geometry parameter m. In Fig. 3 we show similar curves for $m = 0.5$. These can be used as examples to demonstrate the process of symmetric snapping of a bar subject to a load (I) and when there is no load (II). We denote by III the zone of unstable equilibrium of the bar.

For asymmetric snapping ($W_1 \neq 0$) Eqs (0.3) admit of the solution

$$W_0 = \frac{4}{3}\left(1 - \frac{q^*}{\pi^5}\right), \quad k^2 = 4\pi^2 \Leftrightarrow P_{cr\,2}^{\text{Euler}} = \frac{4\pi^2 EJ}{l^2}$$

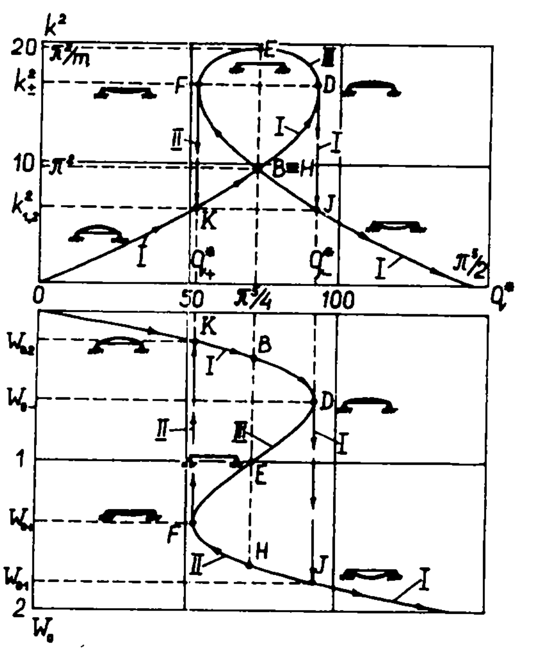

Fig. 3.

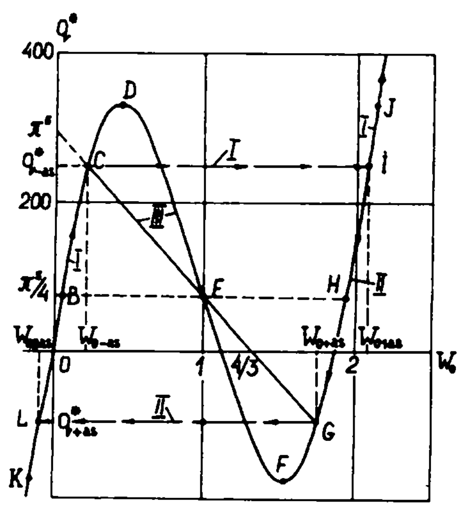

Fig. 4.

For asymmetric snapping it follows that the longitudinal force is equal to the second Eulerian critical force for a rectilinear bar and asymmetric snapping occurs at loads

$$q^*_{\mp as} = \frac{\pi^5}{4}\left(1 \pm 3\sqrt{1-m}\right)$$

with the corresponding deflections

$$W_{0_{\mp as}} = 1 \mp \sqrt{1-4m}$$

$$W_{0_{\pm 1,2 as}} = 1 \pm \frac{1}{2}\left(\sqrt{1-4m} + \sqrt{1+8m}\right)$$

From the condition $q^*_{\mp} = q^*_{\mp as}$ we obtain the boundary between the snapping forms: $m = 2/11$. The functions $q^* = q^*(W_0)$ for symmetric and asymmetric snapping and $m = 0.1$ are shown in Fig. 4. The numbers I and II denote the sections of the deformation curve corresponding to the presence and absence of a load, while III corresponds to sections of symmetric and asymmetric unstable equilibrium state. In Fig. 5 we show graphs of the upper critical load q^*_- and lower critical load q^*_+ for symmetric snapping (Timoshenko, curves I) and asymmetric snapping (Grigolyuk, curves II).

An advantage of the approximate solutions [1, 2, 8] is that one can obtain explicitly the necessary parameters of the problem on the loss of stability of the bar and reveal the mechanism of its behaviour for various values of m.

This enables us to obtain a deeper understanding of the part played by Bubnov's research in the problem under consideration.

It should be observed that in [9, 10] Bubnov considered a parabolic cylindrical panel of infinite length subject to a transverse load, rather than a bar. He obtained all the equations necessary to solve the problem, offering a formal mathematical description of snapping as early as in 1902.

1. A PARABOLIC PANEL SUBJECT TO FINITE DEFLECTION

Consider a cylindrical panel of infinite length whose median surface is traced by a parabola of degree two. It is shown schematically in Fig. 6. We will describe the deformation of such a weakly curved panel assuming that it is clamped along the parallel sides so that the support resists rotation of the edges with stiffness α_M and their coming together during the deformation process with stiffness α_N. In the vertical direction the panel is loaded by uniform pressure q and horizontally by stretching forces N_0, constant along the edges. The panel has a width l, thickness h, and is made from a material with modulus of elasticity E and Poisson's ratio μ.

To describe the equilibrium state of the panel we use the equations

$$\frac{dN}{dx} = 0, \quad D\frac{d^4 w}{dx^4} - N\left(\frac{d^2 w}{dx^2} - \frac{d^2 W}{dx^2}\right) = q \quad (0 \le x \le l) \tag{1.1}$$

where w and x are the deflection of the panel and the distance along the axis connecting the supports, D =

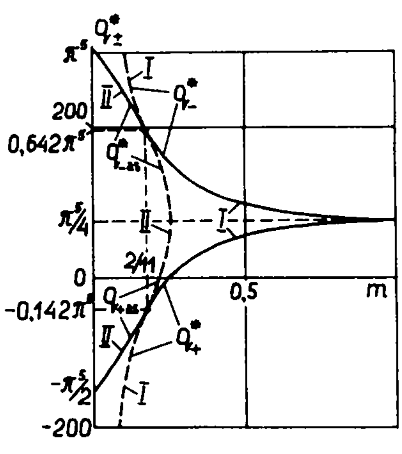

Fig. 5.

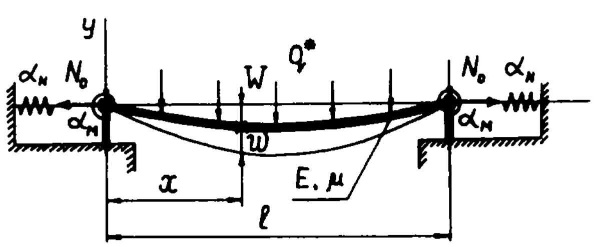

Fig. 6.

$Eh^3/[12(1-\mu^2)]$ is the cylindrical stiffness of the panel, $W = -4H(x/l)(1-x/l)$ is the initial shape of the median surface of the panel, and H is the largest deflection of the median surface from a plane. The relative outward force N in the panel is related to the tangential displacements of the median surface u and the deflection w by a relation which follows from Hooke's law

$$N = \frac{Eh}{1-\mu^2}\left[\frac{du}{dx} + \frac{d^2W}{dx^2}w + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] \tag{1.2}$$

The boundary conditions corresponding to the chosen method of describing the panel are

$$\begin{aligned} &w = 0,\ M = -\alpha_M \frac{dw}{dx},\ N = \alpha_N u + N_0 \quad \text{for } x = 0 \\ &w = 0,\ M = \alpha_M \frac{dw}{dx},\ N = -\alpha_N u + N_0 \quad \text{for } x = l \end{aligned} \tag{1.3}$$

where $M = -Dd^2w/dx^2$ is the specific bending moment in the plate.

The above boundary-value problem has the following solution

$$\begin{aligned} &w^* = \frac{q^* - k^2\lambda}{2k^2}\left\{\frac{\omega}{k}\left[\operatorname{ch}(k\xi) - 1 - \operatorname{th}\frac{k}{2}\operatorname{sh}(k\xi)\right] + \xi - \xi^2\right\} \\ &u^* = \frac{1+\alpha_N^*\xi}{2(2+\alpha_N^*)}I(1) - \frac{1}{2}I(\xi) + \frac{N_0^*}{12(2+\alpha_N^*)}(2\xi - 1) \\ &I(\xi) = \int_0^\xi \left[2\lambda w^* + \left(\frac{dw^*}{d\xi}\right)^2\right]d\xi,\ \omega = \frac{2+\alpha_M^*}{k+\alpha_M^*\operatorname{th}(k/2)} \end{aligned} \tag{1.4}$$

the relation between the external load and inner forces being

$$k^2 - \frac{2N_0^*}{2+\alpha_N^*} = \frac{6\alpha_N^*}{2+\alpha_N^*}I(1) \tag{1.5}$$

Here we have used the following dimensionless variables: the transversal coordinate $\xi = x/l$, the horizontal and vertical displacements of the median surface of the plate $u^* = ul/h^2$ and $w^* = w/h$, the vertical and horizontal distributed loads $q^* = ql^4/(Dh)$ and $N_0^* = N_0 l^2/D$, the outward force parameter $k^2 = Nl^2/D$, the parameter $\lambda = 8H/h$ denoting the initial dimensionless amplitude of the parabolic irregularity of the panel, and the transversal and bending stiffness $\alpha_N^* = \alpha_N l h^2/(12D)$ and $\alpha_M^* = \alpha_m l/D$ of the supported edges of the plate.

Once the integral $I(\xi)$ is computed, (1.5) gives a quadratic equation for the dimensionless vertical transverse load

$$\begin{aligned} &\left[3k\omega^2\frac{\operatorname{sh}k - k}{\operatorname{ch}^2(k/2)} + 48\omega\left(\operatorname{th}\frac{k}{2} - \frac{k}{2}\right) + 2k^2\right]\left(q^* - k^2\lambda\right)^2 + \\ &+4\lambda k^2\left[12\omega\left(\operatorname{th}\frac{k}{2} - \frac{k}{2}\right) + k^2\right]\left(q^* - k^2\lambda\right) + \\ &+\left[8k^6\frac{N_0^*}{\alpha_N^*} - 4k^8\left(\frac{2}{\alpha_N^*} + 1\right)\right] = 0 \end{aligned} \tag{1.6}$$

As a result, the largest deflection of the panel observed along the central line can be determined using (1.4)

$$w_{\max}^* = w^*\left(\frac{1}{2}\right) = \frac{1}{2k^2}\left\{\frac{\omega}{k}\left[\frac{1}{\operatorname{ch}(k/2)} - 1\right] + \frac{1}{4}\right\}\left(q^* - k^2\lambda\right) \tag{1.7}$$

the stresses on the upper and lower sides of the panel along the central line and along the edges being given by

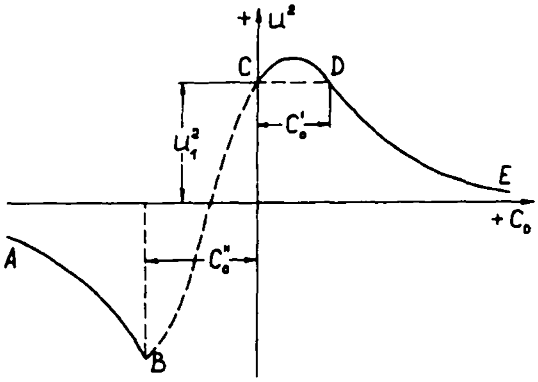

Fig. 7.

$$
\begin{aligned}
\sigma^*\left(\frac{1}{2}\right) &= k^2 \mp \frac{3}{k^2}\left[\frac{\omega k}{\operatorname{ch}(k/2)} - 2\right]\left(q^* - k^2\lambda\right) \\
\sigma^*(0) = \sigma^*(1) &= k^2 \mp \frac{3}{k^2}(\omega k - 2)\left(q^* - k^2\lambda\right)
\end{aligned}
\tag{1.8}
$$

where $\sigma^* = \sigma h l^2/D$ are the dimensionless stresses.

2. BUBNOV'S SOLUTION

We will now compare the above solution of the problem of panel deformation with that obtained by Bubnov. For a parabolic panel with clamped longitudinal edges, using the elastic properties of the longitudinal supports in the horizontal plane he gave the following equation relating the outward force parameter to the vertical load ([9], "Supplement 1. Influence of the curvature of the sheet", formula (40))

$$
\frac{(1-\alpha)^2}{F_2(u)} + \frac{315}{u^3}\alpha(1-\alpha)\left[1 - \chi_2(u)\right] = \left[\frac{64}{35}\left(1-\mu^2\right)^3\left(\frac{p}{E}\right)^2\left(\frac{a}{t}\right)^8\frac{T}{t+T}\right]^{-1}
$$

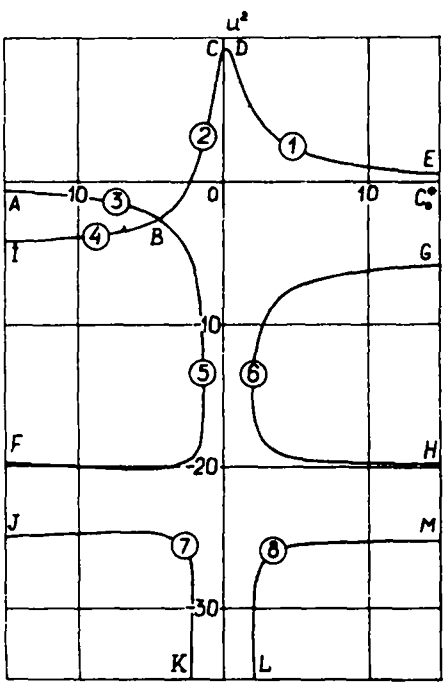

Fig. 8.

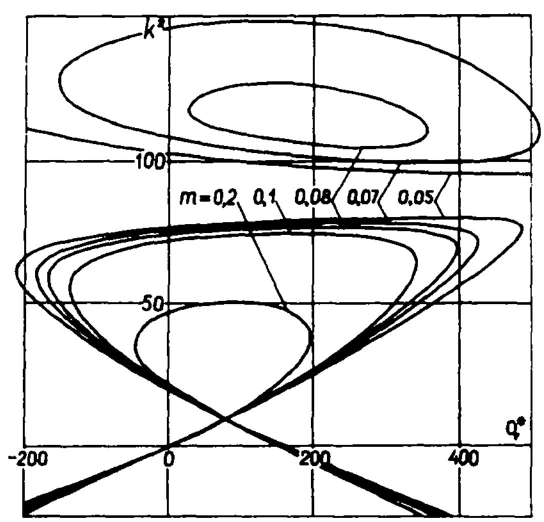

Fig. 9.

where

$$F_2(u)=\frac{1}{945}u^4\left[\frac{5}{12}u+\frac{25}{16}-\left(\frac{3}{4}+\frac{\sqrt{u}}{2\,\mathrm{th}\sqrt{u}}\right)^2\right]^{-1}$$

$$\chi_2(u)=\frac{3}{u}\left(\frac{\sqrt{u}}{\mathrm{th}\sqrt{u}}-1\right)$$

Here p is the distributed vertical load (in this paper it is denoted by q), E and μ are the modulus of elasticity and Poisson's ratio of the material of the plate, t and a are the thickness and half-width of the plate (h and $l/2$), u is the coefficient of the outward force in the plate ($k^2/4$), $T=\alpha_N^* h/[2(1-\mu^2)]$ is the compression–expansion stiffness of the support of the plate in its plane, and α is a parameter defining the curvature of the panel, which can be written as

$$\alpha=\frac{1}{6(1-\mu^2)}\frac{\eta t^3}{a^4}\frac{E}{p}u$$

where η is the largest initial deflection amplitude of the panel (H in the notation of the present paper). As a result, α corresponds to the fraction $k^2\lambda/q^*$ and Bubnov's equations stated above yield the equation

$$\left[3k\frac{\mathrm{sh}\,k-k}{\mathrm{sh}^2(k/2)}+48\left(1-\frac{k}{2}\mathrm{cth}\frac{k}{2}\right)+2k^2\right]\left(q^*-k^2\lambda\right)^2+$$

$$+4\lambda k^2\left[12\left(1-\frac{k}{2}\mathrm{cth}\frac{k}{2}\right)+k^2\right]\left(q^*-k^2\lambda\right)-4k^8\left(\frac{2}{\alpha_N^*}+\frac{1}{1-\mu^2}\right)=0$$

where the notation of the present paper is used. This equation is the same as a limiting case of (1.6) with $N_0^*=0$ and $\alpha_M^*\to\infty$, apart from the last term, which should be written differently: $4k^8(2/\alpha_N^*+1)$. This is a consequence of an inaccuracy by the author. The right-hand side of Bubnov's equation (40) should contain the fraction $T/[t+(1-\mu^2)T]$ in place of $T/(t+T)$.

A generalized solution of the problem under consideration with the inaccuracy removed was given by Bubnov in [10, 11, Section 25]. He stated the following formulae describing the process of deformation of the panel.

The equation relating the outward force parameter to the vertical load

$$\frac{9\varepsilon(1-\varepsilon)}{4u^6}\left\{1-\left[\varkappa+(1-\varkappa)\frac{\mathrm{th}\,u}{u}\right]\varkappa(u)\right\}+$$
$$+(1-\varepsilon)^2F(æ,u)=\frac{1}{K\left(1-\mu^2\right)^2}\left(\frac{E}{p_x^0}\right)^2\left(\frac{h}{2a}\right)^8\left(1-\frac{p_z^0}{p_z}\right) \qquad (2.1)$$

where $\varepsilon=2p_z hc_0/(p_x^1 a^2)$ is the reduced initial irregularity of the panel (c_0 is the largest initial deflection amplitude), which in the notation of the present paper corresponds to $k^2\lambda/q^*$, as does α from [9]. The other variables and functions in this equation can be described as follows:

$$F(æ,u)=(1-æ)U_0+æU_1-æ(1-æ)U_2)$$

$$U_0=\frac{9}{8u^6}-\frac{27}{16}\frac{5(u-\mathrm{th}\,u)-u\,\mathrm{th}^2u}{u^9}$$

$$U_1=\frac{45}{16u^6}-\frac{27}{16}\frac{(u-\mathrm{th}\,u)(u+4\,\mathrm{th}\,u)}{u^8\,\mathrm{th}^2u}$$

$$U_2=\frac{27}{16}\frac{(u-\mathrm{th}\,u)^2}{u^9\,\mathrm{th}^2u}\left(u\,\mathrm{th}^2u-u+\mathrm{th}\,u\right)$$

h and a are the thickness and half-width of the plate (in the present paper they are denoted by h and $l/2$), p_z is the expanding stress in the panel (N/h), p_z^0 is the expanding stress at the edges of the panel (N_0/h), p_x^0 is the uniformly distributed vertical load (q), and K and κ are the dimensionless transverse and bending compression–expansion stiffness of the supports of the panel described by

$$K = \frac{\alpha_N^*}{2+\alpha_N^*(1-N_0^*/k^2)^{-1}}, \quad \chi = \frac{\alpha_M^* \operatorname{th}(k/2)}{k+\alpha_M^* \operatorname{th}(k/2)}$$

in the notation of the present paper. By the outward force parameter u we mean a slightly different variable than in [9]. In the present paper it is denoted by $k/2$, rather than $k^2/4$, as before.

The relationship for the maximum deflection of the panel

$$\xi_{\max} = \left(1-\mu^2\right)(1-\varepsilon)\frac{p_x^0}{E}\frac{(2a)^4}{h^3} f(æ,u)$$

where

$$f(æ,u) = \frac{5}{32}(1-æ)f_0(u) + \frac{1}{32}æ f_1(u)$$

$$f_0(u) = \frac{12}{5u^2}\left(1-\frac{2}{u^2}+\frac{2}{u^2\operatorname{ch}u}\right)$$

$$f_1(u) = \frac{12}{u^2}\left(1-\frac{\operatorname{th}(u/2)}{u/2}\right)$$

These Bubnov's relationships are identical with (1.6) and (1.7) if they are rewritten in the notation of the present paper.

3. DISCUSSION OF BUBNOV'S RESULTS

In [10, 11, Section 25, Fig. 78] Bubnov presented a graph resulting from a numerical study of Eq. (2.1) (the graph is shown in Fig. 7). It was obtained as follows. For specified values of the bending stiffness κ of the supports and the right-hand side of the coupling equation the outward force parameter u was given. The largest initial deflection amplitude c_0 of the panel was determined from the quadratic equation for the reduced curvature ε of the panel. It was observed that "for a negative deflection amplitude the curve consists of two branches, where branch BC corresponds to an unstable equilibrium of the plate, but for deflections whose absolute value exceeds the critical amplitude c_0'' the plate will be stable (branch AB)". At this critical deflection amplitude, if a bending load p_x^0 is applied, then inward stresses p_z will appear equal to the Eulerian ones and the imaginary argument u will reach its limit value ($u = \pi\sqrt{(-1/2)}$ for $æ = 0$, and $u = \pi\sqrt{(-1)}$ for $æ = 1$). Any further increase in p_0^x will change the curvature of the plate to the opposite one, so that the deflection amplitude c_0 will become positive, and the equilibrium of the plate will be stable.

Thus, we can see that Bubnov came close to solving the problem of the snapping of weakly bent plates and flat shells. However, he restricted himself to a qualitative discussion, having presented in his Fig. 78 merely the possible types of behaviour of the panel, rather than the process of loading for a specific panel. Besides, Fig. 78 is just a small part of the true picture.

In fact the behaviour of the panel can be described by infinitely many curves forming a family of lines unbounded in the domain of negative values of the square of the outward force parameter u. This can be seen in Fig. 8, where $c_0^* = c_0/h$ is given as a function of u^2 using Eq. (2.1) for a panel on a hinged support with parameters $K = 1$, $æ = 0$, $\mu = 0.3$, $p^0{}_z = 0$ and $(1-\mu^2)(p_x^0/E)(2a/h)^4 = 30$. Here a version of Eq. (2.1) with u replaced by iu was used for the negative values of u^2.

A similar equation in the notation of [8], in particular for a parabolic bar with a hinged support subject to a uniform vertical load can be written as [12]

$$\frac{1}{2k^6}\left[2+\frac{k^2}{12}+\frac{k-5\sin k}{2k\cos^2(k/2)}\right]\left(q^*-8k^2\right)^2 - \frac{8}{k^3}\left[\frac{\sin k}{k^2\cos^2(k/2)}-\frac{k}{12}-\frac{1}{k}\right]\left(q^*-8k^2\right)+\frac{m}{4}k^2=0 \tag{3.1}$$

where the dimensionless load q^*, the outward force parameter k and the bar geometry parameter m are given by

$$q^* = \frac{ql^4}{EJH}, \quad k^2 = -\frac{Nl^2}{EJ}, \quad m = \frac{4J}{FH^2}$$

Here q is the vertical load, uniformly distributed along the bar, N is the outward force in the bar, l and H is the length and the elevation amplitude of the bar, E is the modulus of elasticity of the material, F is the area of cross-section of the bar and J is its moment of inertia.

The parameters in (3.1), which are denoted by a subscript $[G]$, are related as follows to the parameters in Bubnov's equation (2.1) marked by $[B]$ and the parameters in the equations in the present paper (these are indicated by $[H]$)

$$k^2_{[G]} = -\frac{4}{1-\mu^2}u^2_{[B]} = -12\frac{p_{z[B]}}{E}\left(\frac{2a}{h}\right)^2 = -\frac{k^2_{[H]}}{1-\mu^2}$$

$$\overset{*}{q}_{[G]} = 12\frac{p^0_{x[B]}}{E}\left(\frac{2a}{h}\right)^4\frac{h}{c_0} = \frac{\overset{*}{q}_{[H]}}{1-\mu^2}\frac{h}{H} \tag{3.2}$$

$$m_{[G]} = \frac{1}{3}\left(\frac{h}{c_0}\right)^2 = \frac{1}{3}\left(\frac{h}{H}\right)^2, \quad \varepsilon_{[B]} = -8\frac{k^2_{[G]}}{\overset{*}{q}_{[G]}}$$

Equations (1.6) and (2.1), written for a panel with a hinged support ($\alpha_N^* \to \infty$, $\alpha_M^* = 0$, $N_0^* = 0$ and $K = 1$, $æ = 0$, $p_z^0 = 0$, respectively) in the case when the panel is bent in the opposite direction to the vertical load, reduce to (3.1) only if the factor $(1-\mu^2)$ is omitted when transforming the parameters. This means that the bending and longitudinal stiffness of the bar will change as the parameters are replaced by the cylindrical stiffness and the compression-expansion stiffness of the panel. For the above-mentioned equations to be exactly the same one must use the following relations in place of (3.2)

$$k^2_{[G]} = -4u^2_{[B]} = -12\left(1-\mu^2\right)\frac{p_{z[B]}}{E}\left(\frac{2a}{h}\right)^2 = -k^2_{[H]}$$

$$\overset{*}{q}_{[G]} = 12\left(1-\mu^2\right)\frac{p^0_{x[B]}}{E}\left(\frac{2a}{h}\right)^4\frac{h}{c_0} = \overset{*}{q}_{[H]}\frac{h}{H}$$

$$m_{[G]} = \frac{1}{3}\left(\frac{h}{c_0}\right)^2 = \frac{1}{3}\left(\frac{h}{H}\right)^2, \quad \varepsilon_{[B]} = -8\frac{k^2_{[G]}}{\overset{*}{q}_{[G]}}$$

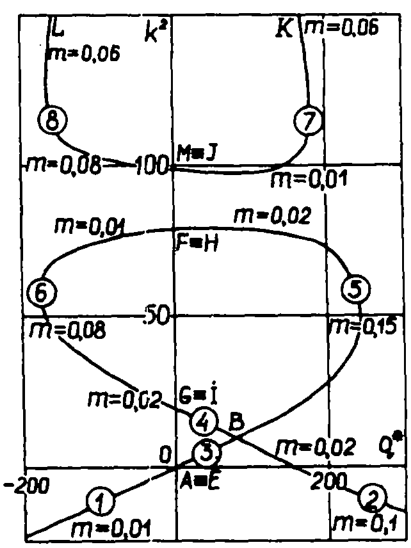

Fig. 10.

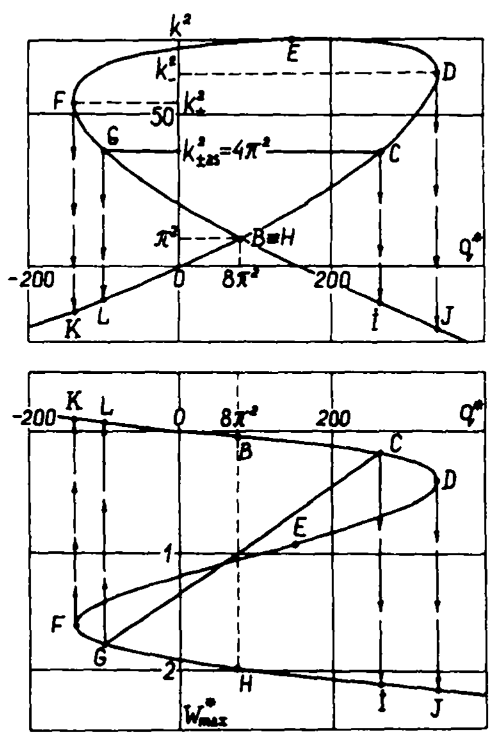

Fig. 11.

It then becomes possible to transform the curves obtained in [8, 12] for the solution of Eq. (3.1) representing a real process of loading an arch or panel into the curve field in Fig. 78 of [10, 11], which shows the possible equilibrium states of various panels, and vice versa. For a uniformly transversely loaded bar (panel) with a hinged support and various values of the geometry parameter m the load curves look as in Fig. 9. If Fig. 8 is transformed into the coordinates in Fig. 9, then Bubnov's curve (Fig. 10) will be the geometric locus of the points describing stable and unstable equilibrium states of a panel related to one another by

$$\frac{\overset{*}{q}{}^{2}_{[G]}}{m_{[G]}} = 432(1-\mu^2)^2\left(\frac{p^0_{x[B]}}{E}\right)^2\left(\frac{2a}{h}\right)^8 = 388800$$

Although Bubnov was the first to develop a mathematical model of the deformation of the weakly curved panel, it should be noted that he failed to represent the stability loss of the panel completely. When discussing the deformation of the panel he made an error in identifying the stable and unstable branches of the loading trajectories, asserting that stability loss of a panel with a hinged support occurs when the outward force parameter is equal to the first Eulerian critical force $k^2_{[G]} = -4u^2_{[B]} = \pi^2$, since "any further increment of the load p^0_x will cause a change in the plate curvature to the opposite one." However, as can be seen in Fig. 11 for a parabolic panel or in Fig. 4 for a sinusoidal bar, this is not so. If the panel is subject to a distributed load on the convex side, its stress–strain state will be represented by a point moving along the loading line from O through B towards C and D as the load increases, the outward force parameter at B being equal to the first Eulerian critical force, but there will be no stability loss. Symmetric stability loss occurs at D with a jump of the loading process to J or, when the value of the outward force parameter k^2 corresponding to the symmetric stability loss is greater than the second critical Eulerian force, asymmetric stability loss will take place at C with a jump to I. However, Bubnov considered B to be a critical point, apparently assuming a jump from B to H.

For asymmetric stability loss the value of the outward force parameter is equal to $4\pi^2$, the second critical Eulerian force for a flat panel [8]. Possibly, had Bubnov analysed the behaviour of the panel in detail without restricting himself solely to the first Eulerian critical force, he would have been led to the idea that the first Eulerian critical force is not critical for a panel and that asymmetric stability loss is possible.

Bubnov's partial study of the behaviour of a parabolic panel did not enable him to understand its behaviour mechanism completely. However, he evidently did not aim to achieve this, but, as an engineer, to solve a practical problem arising in ship building. The parameters of the problem he chose did not reveal the full behaviour pattern of a panel subject to a load. Of course it is now too late to ask if Timoshenko was aware of Bubnov's results [9, 10]. Clearly, he was. It is a different matter whether or not these results inspired him to solve the problem of the buckling of a bar subject to a load. Undoubtedly, this was not so, and the idea of studying the problem occurred to him in relation to the behaviour of curved bimetal strips, an approximate theory of which he developed.

In any case, it is evident that Bubnov was the first to state the Foeppl–Kármán–Marguerre equations for finite deflections of Kármán thin-walled elastic surfaces. His results on the behaviour of circular plates as well as plane and curvilinear panels entitle him to this claim. As regards the problem of the snapping of thin-walled surfaces, it should be observed that Bubnov was the first to obtain a solution of the non-linear problem of the behaviour of a parabolic cylindrical panel, which in principle makes it possible to draw all the necessary conclusions obtained subsequently by others (Timoshenko and Grigolyuk). Unfortunately, Bubnov could not and did not try to use his results to formulate the general problem of shell stability. Timoshenko, who devoted a number of research projects to this theory, did not raise this question either. But Kármán and Tsien's work [6] on the snapping of a spherical dome under uniform radial pressure must have been strongly influenced by Timoshenko's result [2] on the problem of the stability loss of a shell under finite deflection, since the model of a spherical dome is very close to that of a flat bar.

REFERENCES

1. TIMOSHENKO S., Analysis of bimetal thermostats. *J. Optical Soc. Amer. and Rev. of Scient. Instruments* **11**, 3, 233–255, 1925; reprinted in TIMOSHENKO S. P., *The Collected Papers*, pp. 403–421. McGraw-Hill, New York, 1953.
2. TIMOSHENKO S., Buckling of flat curved bars and slightly curved plates, *J. Appl. Mech.* **2**, 1, 17–20, 1935; Reprinted in TIMOSHENKO S. P., *The Collected Papers*, pp. 508–515. McGraw-Hill, New York, 1953.
3. NAVIER C. L. M. H., *Résume des lécons données à l'elablissement des constructions et des machines*, Pt 1, pp. 226–286. Chez Corillan–Gceury, Paris, 1926.
4. NADAI A., *Labile Gleichgewichtslagen stark gebogenen Stäbe, Technische Blätter*, Z. Deutschen Polytechn. Vereins in Böhmen, Prague **47**, 3–4, 137–145, 1915.
5. BIEZENO C. B., Über eine Stabilitätsfrage beim gelenking gelagerten schwach gekrümmten Stabe. *Proc. Koninkl. Akad. Wetenschappen Amsterdam* **32**, 7, 990–994, 1929; Das Durchshlagen eines schwach gekrümmten Stabes, *ZAMM* **18**, 1, 21–30, 1938.
6. KÁRMÁN Th. and TSIEN H.-S., The buckling of spherical shells under external pressure. *J. Aeronaut. Sci.* **7**, 2, 43–50, 1939; reprinted in KÁRMÁN Th. *Collected Works*, Vol. 3, pp. 368–380. Butterworths, London, 1956.
7. KÁRMÁN Th. and TSIEN H.-S., The buckling of cylindrical shells under axial compression. *J. Aeronaut. Sci.* **8**, 303–312, 1941; Reprinted in KÁRMÁN Th., *Collected Works*, Vol. 4, pp. 107–126. Butterworths, London, 1956.
8. GRIGOLYUK E. I., An analysis of the stability of flat arches. *Inzh. Sb.* **9**, 177–200, 1951.

9. BOOBNOFF I. G., On the stresses in a ship's bottom plating due to water pressure. *Trans. Inst. Naval Arch.* **44,** 15–46, 51–52, 1902; Special issue; 1902. See also: BUBNOV I. G., Stresses in ship's bottom plating due to water pressure. *Morskoi Sb.* **311,** 8, 117–141, 1902; **312,** 9, 111–139, **312,** 10, 119–138, **312,** 12, 107–130; Special issue: Tipolitogr., A. E. Vineke, St. Petersburg, 1904; reprinted in BUBNOV I. G., Stresses in ship's bottom plating due to water pressure, in *Proceedings in the Theory of Plates*, pp. 11–100. Gostekhizdat, Moscow, 1953.
10. BUBNOV I. G., *Mechanics of Ship Building*. Tipografiya Morsk. Min., St Petersburg, Pt 2, 331–640, 1914.
11. BUBNOV I. G., *Papers on Plate Theory*. Gostekhizdat, Moscow, 1953.
12. GRIGOLYUK E. I. and ANDRIANOV N. N., The non-linear statistical behaviour of flat bars. In *Some Applied Problems in the Theory of Plates and Shells*, pp. 3–83. Izd. Moscov. Gos. Univ., Moscow, 1981.

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